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A note on the method of steepest descents with a remark on T. Ljunggren's paper "Contributions to the theory of diffraction of electromagnetic waves by spherical particles"[1]

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In his theory of diffraction of electromagnetic waves with the wave-length λ around a sphere with the radius a and relative refraction index n, Ljunggren evaluates a large number of integrals of the form

$$I(\varrho) = \int_b^c g(\tau)\, e^{i\varrho G(\tau)}\, d\tau \tag{1}$$

by means of the method of steepest descents for large values of the parameter $\varrho = 2\pi a/\lambda$. In this method[2], in its pure form, the slowly varying functions $G(\tau)$ and $g(\tau)$ are developed as Taylor polynomials around certain decisive points, namely the limits of integration and the saddle points τ_0, where at least $G'(\tau_0) = 0$. Here we wish to point out that, if the expression for a saddle point has a complicated character, depending e.g. on one or two parameters, it can sometimes be convenient to develop G and g around another fix-point $\bar{\tau}_0$ with simpler properties, which is chosen in the *immediate neighbourhood* of the saddle point under consideration.[3] If this saddle point has the multiplicity m, the approximate polynomial for $G(\tau)$ has still to be of at least the degree $(m + 1)$, and one has further to choose a corresponding integration contour. The slight loss of accuracy connected with this modification is often without any importance and can, if desired, be compensated by considering even higher terms in the asymptotic expansion of the integral (1).

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[1] T. LJUNGGREN, Arkiv f. mat., astr. o. fysik *36* A, n:o 14 (1948); Arkiv f. fysik *1*, n:o 1 (1949); in the following quoted as L I and L II, respectively.

[2] Developed by B. RIEMANN, Ges. Math. Werke, p. 400 (1876); Lord KELVIN, Phil. Mag. *23*, p. 252 (1887); P. DEBYE, Math. Ann. *67*, p. 535 (1909); Münch. Sitz. Ber. *40*, n:o 5 (1910); L. BRILLOUIN, Ann. d. l'Ecole Norm. Sup. *33*, p. 17 (1916); E. HOGNER, Thesis, Uppsala 1925; F. POLLACZEK, Ann. Phys. *2*, p. 991 (1929); H. G. BAERWALD, Ann. Phys. *6*, p. 295 (1930); V. D. CORPUT, Comp. Math. *3*, p. 328 (1936); and others.

[3] A variation of this idea has previously been employed e.g. in the case of two closely situated simple saddle points, where it has been recommended to develop around the point τ_{00}, where $F'(\tau_{00}) = 0$. The integral can then be expressed in terms of Airy integrals. Cf. J. W. NICHOLSON, Phil. Mag. *19*, p. 247 (1910); L. BRILLOUIN, *loc. cit.*; E. HOGNER, *loc. cit.*; F. POLLACZEK, *loc. cit.*; H. JEFFREYS, Phil. Mag. *33*, p. 451 (1942); T. LJUNGGREN, L I, p. 19 (1948).

Let us consider the case when a simple saddle point is situated near to one of the limits of integration[1], say $\tau = c$. Developing G and g around the point $\tau_0 = c$, we obtain easily the following expression for the main contribution from these two decisive points to the integral (1):

$$
\begin{aligned}
I(\varrho) &= g_c\, e^{i\varrho F_c} \sqrt{\frac{2}{\varrho F_c''}} \cdot \eta(\alpha) - i\, g_c' \frac{1}{\varrho F_c''} e^{i\varrho F_c} \{1 - 2\, i\, \alpha\, \eta(\alpha)\}, \\
\alpha &= -F_c' \sqrt{\frac{\varrho}{2 F_c''}}, \qquad \eta(\alpha) = \int_0^{\infty} e^{i(-2\alpha u + u^2)}\, du.
\end{aligned}
\tag{2}
$$

Since the function $\eta(\alpha)$ can directly be expressed in the ordinary Fresnel integrals, the contribution from a simple saddle point near to a limit is essentially characterized by these integrals.[2]

The purpose of this note is to show that some of the calculations carried out by LJUNGGREN (L II, p. 3–8) can be essentially simplified by using the formula (2). Let us investigate Ljunggren's integral

$$
\begin{aligned}
&I(\varrho, n, \Theta) = \\
&= \frac{4 n \varrho^2}{\pi} \left(1 + \frac{\Theta^2}{3}\right) \sum_{N=-\infty}^{+\infty} (-1)^N \int_0^{\frac{\pi}{2}} d\Psi \int_0^{\frac{\pi}{2}} \frac{\sin^2 \tau_1 \sin \tau_2 \cos \tau_1}{(n \sin \tau_1 + \sin \tau_2)^2} \{e^{i\varrho G_+^{(1)}} + e^{i\varrho G_-^{(1)}}\}\, d\tau_1, \\
&G_{+,-}^{(1)} = \cos \tau_1 \cdot (-2\pi N - 2\tau_1 + 2\tau_2 \pm \Theta \cos \Psi) + 2 \sin \tau_1 - 2 n \sin \tau_2, \qquad (3) \\
&\cos \tau_2 = \frac{1}{n} \cos \tau_1,
\end{aligned}
$$

(L II, formulae 5 and 6) for large values of ϱ. The main contribution from the series in N arises from the term $N = 0$, where the inner integral has a simple saddle point in the neighbourhood of the upper limit of integration $\tau_1 = \pi/2$ (above or below this point). Using Brillouin's method, Ljunggren develops his functions G and g around the rather complicated saddle points, and the calculations can then be performed only with a considerable amount of labour. His final expressions (L II, 25 and 25′) are complicated, and, in order to get a survey of his result, Ljunggren then summed the infinite series involving J_0 by numerical summation for some typical values of the variable.

Let us here first consider the inner integral for $N = 0$:

$$
I_{\text{inner}} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \tau_1 \sin \tau_2 \cos \tau_1}{(n \sin \tau_1 + \sin \tau_2)^2} \{e^{i\varrho G_+^{(1)}} + e^{i\varrho G_-^{(1)}}\}\, d\tau_1. \tag{4}
$$

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[1] Cf. the treatment of the same problem given by BRILLOUIN, *loc. cit.*

[2] Cf. BAERWALD, *loc. cit.*

Developing the functions G and g around the upper limit $\tau_1 = \pi/2$ with $\tau_2 = \pi/2$, we get

$$\begin{gathered} g_c = 0, \qquad g_c' = -\frac{1}{(n+1)^2}, \\ G_c = -2(n-1),\ G_c' = \mp \Theta \cos \Psi,\ G_c'' = 2\frac{n-1}{n}, \end{gathered} \tag{5}$$

and

$$\alpha = \pm k \cos \Psi, \tag{6}$$

with the new notation

$$k = \frac{1}{2} \varrho^{1/2} \Theta \sqrt{\frac{n}{n-1}}. \tag{7}$$

According to the simplified formula (2), we then directly obtain

$$\begin{aligned} I_{\text{inner}} &= \frac{i n}{2\varrho (n-1)(n+1)^2} e^{-2i\varrho(n-1)} \{2 - 2ik\cos\Psi \cdot \eta(k\cos\Psi) + \\ &\qquad + 2ik\cos\Psi\cdot\eta(-k\cos\Psi)\} = \\ &= \frac{i n}{2\varrho(n-1)(n+1)^2} e^{-2i\varrho(n-1)} \left\{1 - 2k\cos\Psi \int_0^{+\infty} e^{iu^2} \sin(2ku\cos\Psi)\,du\right\}. \end{aligned} \tag{8}$$

In order to carry out the outer integration, we introduce the function

$$\begin{aligned} \chi &= \frac{2}{\pi}\int_0^{\pi/2} \left\{1 - 2k\cos\Psi \int_0^{+\infty} e^{iu^2}\sin(2ku\cos\Psi)\,du\right\} d\Psi = \\ &= 1 - 2k\int_0^{+\infty} e^{iu^2} J_1(2ku)\,du = 1 - (1 - e^{-ik^2}) = e^{-ik^2}, \end{aligned} \tag{9}$$

where the integral in the second line has been transformed by changing the path of integration and developing the Bessel function J_1 in a power series. Hence we get for (3) the final result

$$I(\varrho, n, \Theta) = 2i\varrho e^{-2i\varrho(n-1)} \left(1 + \frac{\Theta^2}{3}\right) \frac{n^2}{(n-1)(n+1)^2} \chi. \tag{10}$$

where Ljunggren's function χ now can be expressed by the simple analytical formula

$$\chi = e^{-i\frac{n}{4(n-1)}\varrho\Theta^2}. \tag{11}$$

It is remarkable that, for very small angles Θ, Ljunggren has obtained the approximate formula (L II, 27′)

$$\chi \approx 1 - i\frac{n}{4(n-1)}\varrho\Theta^2, \tag{12}$$

and that his numerical results according to (L II, 25′) indicate the validity of the relation $|\chi| \approx 1$ (L II, 48 and 49).

Finally, we will give a comparison between Ljunggren's numerical values of χ and the values obtained from eq. (11). It is suitable to tabulate χ as a function of the variable

$$\beta = \frac{n}{8(n-1)} \varrho\, \Theta^2, \qquad (13)$$

(cf. the slightly different choice of β in L II 21), and the following table gives a survey of χ for some typical values of this variable:

β	$\chi = e^{-2i\beta}$ Analytically according to (11)	χ Numerically according to L II 25′
0.1	$0.980-0.199\,i$	$0.982-0.199\,i$
0.3	$0.825-0.565\,i$	$0.826-0.567\,i$
0.5	$0.540-0.841\,i$	$0.540-0.841\,i$
0.7	$0.170-0.985\,i$	$0.169-0.985\,i$
0.9	$-0.227-0.974\,i$	$-0.231-0.971\,i$
1.1	$-0.589-0.808\,i$	$-0.595-0.795\,i$
1.3	$-0.857-0.516\,i$	$-0.864-0.490\,i$
1.5	$-0.990-0.141\,i$	$-0.993-0.097\,i$
1.6	$-0.998+0.058\,i$	$-0.997+0.116\,i$

The agreement is excellent, and each of the results may be considered as a check of the other. Consequently the small loss of accuracy associated with the use of the simplified formula (2) must here be without any practical importance; on the other hand we have gained a considerable simplification of the final result.

This remark was first given by the author on March 19th, 1949, in connection with the public discussion of T. Ljunggren's thesis for the degree Doctor of Philosophy. I am indebted to Dr. LJUNGGREN for placing his numerical material at my disposal.

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